Higher monoidal structures in representation theory.

This mini-course provides an introduction to higher representation theory, from a point of view of Tannaka duality. We start from classical representation theory of groups to build up the Tannaka formalism that allows to translate algebraic structures into categorical ones. Richer monoidal structures on representation categories, and relations between such, lead to more involved underlying algebraic structures as well. In order to understand these algebraic and categorical structures, a "2-dimensional" version of representation theory comes into play. From there, we aim to provide a view on some recent developments in the field and remaining challenges.

We expect participants to have some basic knowledge in algebra, representation theory and category theory.

Lecture 1: Monoidal structure on representation categories.
Our starting point is the classical theory of representations of (algebraic) groups. Motivated by the fact that representations of groups allow for tensor products and duals, we discuss how via the Tannakian formalism, monoidal structures on a category of (co)modules over a fixed (co)algebra translate on additional structures on this (co)algebra: bialgebras and Hopf algebras.

Lecture 2: Braided categories, centers, Yetter-Drinfeld modules and doubles.
We explain how imposing an additional braided structure on the monoidal category of modules over a bialgebra results in a quasi-triangular structure on this bialgebra. A canonical way to build a braided monoidal category from a given monoidal category is by means of its Drinfeld center. In case of categories of modules, the Drinfeld center can be interpreted as the category of Yetter-Drinfeld modules. Under some finiteness conditions, the latter in turn is isomorphic to the category of modules over the Drinfeld double bialgebra, which therefore has a quasi-triangular structure.

Lecture 3: Actegories and comodule algebras.
In spirit of Tannaka duality, monoidal categories can be viewed as a categorical analogue of groups. As a next step, one can consider the representations of a monoidal category, which are called actegories (or somewhat confusingly, "module categories"), which could be thought of as a "2-dimensional" version of classical representations. Applying the Tannaka duality another time, we see that such actegories in turn correspond to representation categories of comodule algebras (or module coalgebras). We study furthermore how a braided structure on the initial monoidal category influences this correspondence.

Lecture 4: Monoidal functors between representation categories: bi-Galois coobjects.
So far, we considered only one Hopf algebra or monoidal category at once. Our next aim is to understand interrelations between several of those, that is, we study monoidal functors and equivalences. In case of categories of modules over bialgebras, monoidal functors correspond to module coalgebras and monoidal equivalences to bi-Galois coobjects. We explain how such objects can be organized into a groupoid and naturally lead to the structure of a Hopf category.

Lecture 5: The center of bi-actegories and bi-categorical perspective.
We now have all ingredients to introduce braided structures and the center

construction for actegories over monoidal categories. Analogously to what we have seen in Lecture 2, such centers can be described in terms of a generalized Yetter-Drinfeld modules. By varying over all possible Hopf algebras, we obtain a rich bicategorical structure that provide a 2-dimensional analogue of Turaev's group-crossed braided monoidal categories. Time permitting, we discuss the link of such structures to Homotopy Quantum Field Theories.

References:

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